# Key Concepts in Descriptive Statistics

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    - Is some information arbitrary?

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- However, we shall discover that this "simplicity" allows us to see the concepts underlying some familiar statistical formulas.
- This discovery is typical of much of statistics: complex-looking formulas can mask some powerful yet simple underlying concepts.

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- Here is a diagram of two lists.

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- Often, we can use a simple equation to indicate a listwise operation.
- For example,

$$Y = X + 2 \tag{1}$$

means "add 2 to every number in the $X$ list to create a new list, called $Y$.

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- Can you describe what you see?
- What changed about the numbers?
- What did not change?
- What about subtracting a number from every number in a list?

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- Suppose the listwise transformation formula is

$$Y = 2X \tag{2}$$

```
1 2   4
  2   4      8
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- Asking questions about invariance can sometimes produce profound insights.
- For example, Einstein mused that "Relativity Theory" might better have been called "Invariance Theory," since fundamentally, it dealt with what remained invariant in the space-time continuum.

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  - Listwise addition or subtraction moves the numbers as a group, as though they were mounted on a rigid stick, and slid to the left or right.
  - Listwise addition does not change any of the distances between numbers.
  - Listwise multiplication or division *by a positive number* can move the numbers as a group, but also causes them to "fan in" or "fan out."

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- These new numbers contain all the information in the original list, and the original list can be reconstructed perfectly from these new numbers.
- However, by recasting the information in this new form, we can get a better “handle” on what information is really contained in a list of numbers, and what the invariance properties are.

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- But it turns out that these principles hold for *any reasonable measures* of these three quantities.
- These principles will allow us to say all kinds of interesting things about statistics while doing virtually no mathematics!

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- If the number of numbers ($N$) is odd, the median is simply the middle value.
- If the number of numbers is even, we will define the median as the average of the two middle values, i.e., a point halfway between the two middle values on the number line.

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- We will use the letter $S$ (for spread) to stand for the range.

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- The resulting *Shape parameters* are $1, 1, 2, 0.5$

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    - $2, 4, 6, 12, 20$
    - $13, 16, 19, 28, 40$

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  - What is the effect of listwise addition(subtraction) on Location, Spread, and Shape?
  - What is the effect of listwise multiplication (division) by a positive number on Location, Spread, and Shape?

# The Vulnerability Box

- Let's present our results in a summary table I'll refer to as The Vulnerability Box. (Einstein would probably call it the Invariance Box.)

| Operation | | Effect on | |
|---|---|---|---|
| | Location | Spread | Shape |
| $+$ | $+$ | | |
| $-$ | $-$ | | |
| $\times$ | $\times$ | $\times$ | |
| $\div$ | $\div$ | $\div$ | |

# Exploiting the Vulnerability Box

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- Tracking changes in a list of numbers
- Rescaling numbers (Changing the Location and/or Spread without affecting Shape).
- Deriving statistical theory.

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- The Vulnerability Box can be used to examine any operation that can be expressed as a sequence of listwise additions, subtractions, multiplications, and/or divisions.

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- Suppose you have a list of $X$'s with a Location of 75 and a Spread of 20. What will the Location and Spread become if you convert them to $Y$'s with this formula?

$$Y = 4\left(\frac{2X + 30}{20}\right) + 2 \tag{3}$$

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- How about Spread?
- What about Shape?

# Solutions

- Location starts at 75. All listwise operations affect location: Multiply by 2 ($75 \times 2 = 150$ ), Add 30 (180), Divide by 20 (9), Multiply by 4 (36), Add 2 (38).

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- Spread starts at 20. Only multiplication or division listwise operations affect spread: Multiply by 2 (40), Add 30 (no change 40), Divide by 20 (2), Multiply by 4 (8), Add 2 (no change 8).
- Shape will stay the same.

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- What can I do?

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- Assuming our grades are at an interval level of measurement, we are now going to adjust the Location and Spread to values that are "culturally appropriate."

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- Let's see how this works.

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- We want to adjust the Spread first. It is currently 40, and we want 20. The lesson of the Vulnerability Box is that multiplication "comes straight through in the Spread and Location."
- So, if we multiply all the numbers by $1/2$, we will multiply both the Spread and Location by $1/2$. So if we started with numbers with $M = 50$ and $S = 40$, we will now have numbers with $M = 25$ and $S = 20$, and this set of numbers will have the same Shape as when we started.

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- So, if we multiply all the numbers by $1/2$, we will multiply both the Spread and Location by $1/2$. So if we started with numbers with $M = 50$ and $S = 40$, we will now have numbers with $M = 25$ and $S = 20$, and this set of numbers will have the same Shape as when we started.
- We can then adjust the Location to $M = 80$ by adding 55 to all the numbers. This will not change the Spread, and will result in a set of numbers with the same Shape as the original numbers, but a Location of 80 and a Spread of 20.

# Rescaling Numbers

## Example (Rescaling Numbers)

We start with 30,50,70. By our current primitive measures of Location and Spread, these three evenly spaced numbers have a Location of 50 and a Spread of 40.

After multiplying by $1/2$, we have three numbers 15,25,35 that have the desired Spread of 20, and are still evenly spaced.

Notice that the Location has changed, to $(1/2) \times 50 = 25$. We want it to be 80. So we must add the difference between where we are (25) and what we want (80), i.e., $80 - 25 = 55$.

After adding 55, we have 70,80,90.

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- The fundamental idea behind rescaling is to:
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- We can turn these informal ideas into a formal “prescription” or “set of formulas” for accomplishing linear rescaling.

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- If we then add $M_y - aM_x$, the Location will become $aM_x + (M_y - aM_x) = M_y$, and we will have accomplished our objective.
- So the "prescription for linear rescaling is $Y = aX + b$, where $a = S_y/S_x$, and $b = M_y - aM_x$.

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  - What will be the Location, Spread, and Shape of the new numbers?

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- We can deduce the Location and Spread of the numbers by simply applying the Vulnerability Box rules.
- We start with $M_x$ and $S_x$, and subtract $M_x$ from all the numbers. The Vulnerability Box tells us that this will not affect the Spread, which will stay at $S_x$, while the Location will change to $M_x - M_x = 0$.

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- We now have numbers with a Location of 0 and a Spread of $S_x$. If we divide them all by $S_x$, we will divide both the Location and Spread by $S_x$. The result is that the Location will be $0/S_x = 0$, and the Spread will be $S_x/S_x = 1$.

# Deriving Statistical Theory

- We have proven that for any list of numbers with non-zero spread, the "$Z$-score transformation" produces numbers with the same Shape as the original numbers, but a Location of 0 and a Spread of 1.

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- Let's consider the Spread first. It is currently 1. We want it to be something else. What do we need to do to the numbers to change the Spread to that "Something Else"? (Answer from C.P.)
- Will the Location have changed? No, it is still zero. So what do we need to do to the numbers to adjust the Location to something else? (Answer from C.P.)

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- Let's try to get a conceptual handle on what that means.
- First of all, in an algebraic sense, we might say that the result is obvious.
- Algebraically, if

$$Z_x = \frac{X - M_x}{S_x} \tag{5}$$

then, of course

$$X = S_x Z_x + M_x \tag{6}$$

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- What did I mean by that? (C.P. and Demo)

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- So when I said "linear rescaling" in the previous slide, I simply meant any sequence of additions, subtractions, multiplications, or divisions by positive numbers.

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- Remember that all the properties of the Vulnerability Box and $Z$-scores hold for any reasonable measures of Location, Spread, and Shape, so long as you keep the definition consistent within the discussion.

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- By these new definitions, the set of $X$ numbers 70,80,90 has Location 80 and Spread 10.

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- The scores have changed, but the $Z_y$ scores are the same as the $Z_x$ scores!
- For example, the first $Y$ score is 68, and it has a $Z$-score of $(68 - 79)/11$, or $-1$.

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- So if two lists of numbers are the same length and have the same Shape, then if we linearly transform them to have the same metric (i.e., Location and Spread), then the two lists will be made identical.
- This fact is commonly exploited to equalize scores across different sections of a course.

# Linear Transformation Rules Revisited

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- These results immediately follow from our Vulnerability Box results, since $a$ accomplishes multiplication (or division) and $b$ accomplishes addition (or subtraction).

- Consider the equations from the previous slide.

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$$\begin{aligned} a &= \frac{S_y}{S_x} \\ b &= M_y - a M_x \end{aligned}$$

- Notice that we actually deduced these formulas earlier in this module by simply expressing our Vulnerability Box rules in mathematical notation.

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    3. So long as the multiplier/divisor is positive, none of the four basic arithmetic operations affect Shape.
    4. In algebraic notation, if $Y = aX + b$, then $M_y = aM_x + b$, $S_y = |a|S_x$.

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  1. Informally adjust the Spread with multiplication, then adjust the Location with addition.
  2. Convert the $X$ scores to $Z$ scores first, then multiply by the desired Spread and add the desired Location.
  3. Use the linear transformation rule $Y = aX + b$, where $a = S_y/S_x$, and $b = M_y - aM_x$.