

Suggested Answers to Review Problems Probability Theory

1. Let $W2$ = “the second marble is white.” Let $W1$ = “the first marble is white.”

This problem makes significant use of two facts which are of considerable importance in solving basic problems in conditional probability.

The first fact concerns joint events, and was dramatized in class when we discussed the basic definitions of conditional probability and independence in the context of the simultaneous throwing of a coin and a die. In that lecture I pointed out that if two processes occur together, any event on one process can be partitioned into a set of intersections on all the possible events on the other process.

In the context of that lecture, when we were throwing a die and tossing a coin simultaneously, I pointed out that

$$1 = (1 \cap H) \cup (1 \cap T)$$

and

$$H = (H \cap 1) \cup (H \cap 2) \cup (H \cap 3) \cup (H \cap 4) \cup (H \cap 5) \cup (H \cap 6)$$

In the context of the present problem, this means we can write

$$\begin{aligned} \Pr(W2) &= \Pr(W2 \cap \overline{W1}) + \Pr(W2 \cap W1) \\ &= \Pr(W2 \mid \overline{W1}) \Pr(\overline{W1}) + \Pr(W2 \mid W1) \Pr(W1) \end{aligned}$$

The last equation uses another key fact we developed in class, i.e., that $\Pr(A \cap B) = \Pr(A \mid B) \Pr(B) = \Pr(B \mid A) \Pr(A)$.

Now, when we go to draw the first marble, there are x white marbles and y nonwhite marbles to be drawn. Hence,

$$\Pr(W1) = \left(\frac{x}{x+y} \right)$$

and

$$\Pr(\overline{W1}) = \left(\frac{y}{x+y} \right)$$

When we add a marble to the second container, this means we then have $z + v + 1$ marbles in the container. If the first marble was white, then we have added a white marble to the z white marbles already there. Hence

$$\Pr(W2) = \left(\frac{z}{z+v+1} \right) \left(\frac{y}{x+y} \right) + \left(\frac{z+1}{z+v+1} \right) \left(\frac{x}{x+y} \right)$$

2. Once the first tube has been found to be good, there are 9 possible tubes remaining. All are equally likely to be the second tube. Five are good tubes. Hence, the probability that the second tube is good is $5 / 9$.
3. Let $B =$ "The Bad (two sided) coin is selected." Let $4H =$ "Four Heads are thrown in 4 tosses."

If the coin is the two sided one, then the event $4H$ will always occur when the coin is tossed 4 times. Hence $\Pr(4H | B) = 1$. On the other hand, if one of the two ordinary coins is selected, we know (using either the binomial distribution or the multiplicative rule of probability) $\Pr(4H | \overline{B}) = \frac{1}{16}$.

Since only one of the three coins is bad,

$$\Pr(B) = \frac{1}{3}, \Pr(\overline{B}) = \frac{2}{3}$$

Moreover, by a logic similar to what we used in the first problem, we know that

$$4H = (4H \cap B) \cup (4H \cap \overline{B})$$

whence

$$\begin{aligned} \Pr(B | 4H) &= \frac{\Pr(4H \cap B)}{\Pr(4H)} \\ &= \frac{\Pr(4H \cap B)}{\Pr((4H \cap B) \cup (4H \cap \overline{B}))} \\ &= \frac{\Pr(B \cap 4H)}{\Pr(4H | B) \Pr(B) + \Pr(4H | \overline{B}) \Pr(\overline{B})} \\ &= \frac{\Pr(4H | B) \Pr(B)}{\Pr(4H | B) \Pr(B) + \Pr(4H | \overline{B}) \Pr(\overline{B})} \\ &= \frac{(1)(1/3)}{(1)(1/3) + (1/16)(2/3)} \\ &= \frac{8}{9} \end{aligned}$$

4. Let X_1 be the outcome for the first person, and X_2 be the outcome for the second person.

Let A be the event of interest. This event can occur only if both persons get either 0,1,2, or 3 heads.

Hence

$$\begin{aligned}
\Pr(A) &= \Pr([(X1 = 0) \cap (X2 = 0)] \cup [(X1 = 1) \cap (X2 = 1)] \\
&\quad \cup \dots \cup [(X1 = 3) \cap (X2 = 3)]) \\
&= \Pr([(X1 = 0) \cap (X2 = 0)]) + \Pr([(X1 = 1) \cap (X2 = 1)]) + \\
&\quad \dots + \Pr([(X1 = 3) \cap (X2 = 3)]) \\
&= \Pr(X1 = 0) \Pr(X2 = 0) + \Pr(X1 = 1) \Pr(X2 = 1) \\
&\quad + \Pr(X1 = 2) \Pr(X2 = 2) + \Pr(X1 = 3) \Pr(X2 = 3)
\end{aligned}$$

The probabilities for getting 0,1,2 or 3 heads can be computed easily from the $\mathcal{B}(3, .5)$ binomial distribution. The probabilities are $1/8$, $3/8$, $3/8$, and $1/8$, respectively.

Consequently,

$$\begin{aligned}
\Pr(A) &= (1/8)(1/8) + (3/8)(3/8) + (3/8)(3/8) + (1/8)(1/8) \\
&= 20/64 \\
&= 5/16
\end{aligned}$$

5. The only way this can occur is if there are exactly n heads in $2n$ tosses. This probability can be calculated directly from the $\mathcal{B}(2n, 1/2)$ distribution.

Hence, the result is

$$\binom{2n}{n} (1/2)^n (1/2)^n = \frac{\binom{2n}{n}}{2^{2n}}$$

6. The probability that no item is chosen more than once is the probability that all items are different. We solve this using the general approach

$$\Pr(A) = \frac{n_A}{n_\Omega}$$

This is simply

$$\frac{{}_N P_r}{N^r}$$

7. There are $\binom{10}{2}$ ways of selecting 2 chips from 10. Only 4 of them (the pairs $(9,1), (8,2), (7,3), (6,4)$) result in a sum of 10. Hence the probability of the event is

$$\frac{4}{\binom{10}{2}} = \frac{4}{45}$$

8. There are at least two ways to approach the problem. One uses the counting principle. There are $\binom{52}{5}$ ways of dealing a poker hand. We can enumerate the total number of ways of dealing a flush as follows. First, we must choose a suit, and there are 4 ways of doing this. Then, for each suit, there are $\binom{13}{5}$ ways of selecting the 5 card hand. Hence, our answer is

$$\frac{4\binom{13}{5}}{\binom{52}{5}}$$

There is an alternative way. It is to use the general rule for sequences. When we go to deal the first card, any card will leave intact our chance for obtaining a flush. Once the first card is drawn, there are only 12 cards left (out of 51) which will leave intact the probability of a flush. If one of them is drawn, then there are only 11 (out of 50) cards remaining which will continue the flush, etc.

Hence another way of writing the answer is

$$(1) \frac{12}{51} \frac{11}{50} \frac{10}{49} \frac{9}{48}$$

9. Using the general rule for the number of sequences, we compute the answer as, simply, $(5)(4)(6) = 120$.
10. In order to reach the 4th trial, we need failures on the first 3 trials. The probability of this happening, since the trials are independent, is $.8^3 = .512$.
- 11.

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - \Pr(A) \Pr(B) \end{aligned}$$

since the events are independent. Hence the answer is

$$(.5) + (.6) - (.5)(.6) = .80$$