Sequential Probabilities, Counting Rules, and Combinatorics

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1. Goals for this Module

In this module, we will

1. Develop a general strategy, called the "keep it alive sequential approach," that can be used for a wide variety of problems
2. Work several examples of the use of the "keep it alive" strategy.
    (a) Smoking tables
    (b) Poker hands
    (c) Epidemiology
    (d) Rolling the dice
    (e) Finding matching numbers in a phone book
    (f) Finding matching birthdays in a group of people.
3. Define and discuss "counting rules"
    (a) The general rule for the number of sequences
    (b) Permutations
    (c) Permutations with selection
    (d) Combinations
4. Revisit some earlier problems using counting rules

2. The Keep It Alive Sequential Strategy

In the last class, we learned an important rule for computing probability of a sequence

Corollary 2.1 *The probability of the sequence of events $A_1A_2A_3\cdots A_N$ is the product of the probabilities of events at each point in the sequence conditional on everything that happened previously, i.e.,*

$$\begin{aligned}\Pr(A_1A_2A_3\cdots A_N) &= \Pr(A_1)\Pr(A_2|A_1)\\ &\quad\times\Pr(A_3|A_1A_2)\\ &\quad\times\Pr(A_4|A_1A_2A_3)\\ &\quad\times\cdots\Pr(A_N|A_1A_2\cdots A_{N-1})\end{aligned}$$

You will see many applications for this rule if you recognize its full potential. The key is to realize that the events A_i at each point in the sequence can be defined as *any outcome that "keeps alive" the event of interest*, with the corresponding conditional probability defined as *the probability that the event of interest will be kept alive.* A few examples will suffice to demonstrate this.

Example 2.1 ***(Estimating the Need for Smoking Tables)*** *Several years ago, a movement started in the city in which I was living to convert all restaurants to non-smoking environments. As a first step, all establishments were required to have designated "Non-Smoking" areas. A key question for restaurant owners was precisely what proportion of their tables to allocate to the Smoking and Non-Smoking sections. I couldn't help but notice that, in the restaurants I frequented, it seemed like there were very few Non-Smoking tables. Since only about 1/3 of the general population of adults smoked at the time, this seemed really unfair. But was it? What is the probability that a group of 4 people arriving at a restaurant will require a smoking table?*

Solution 2.1 *Of course, we cannot answer this question definitively without a great deal of research. Like many "real world" problems, there are many complexities to deal with. However, we can make some simplifying assumptions and create a simple probability model that will provide us with some important insights into this question. Suppose, for the sake of simplicity, we assume the following: (a) People arrive at the restaurant in groups of 4; (b) People arrive independently with respect to smoking behavior, i.e., smokers do not "cluster in groups." (This assumption is almost certainly false to a degree.), (c) If at least one person in a group of 4 is a smoker, then that group will require a Smoking table. Define S as the event that a randomly selected person is a smoker. Under those assumptions, and if* $\Pr(S) = 1/3$, $\Pr(\overline{S}) = 2/3$, *we can compute the probability of a group requiring a smoking table using the keep it alive strategy, and by recalling the 2nd Theorem of probability, i.e., that* $\Pr(A) = 1 - \Pr(\overline{A})$. *We will compute the probability of* $\overline{A}$, *then subtract it from* 1 *to obtain the answer. The only way a Non-Smoking table can be selected is if all* 4 *people who form the group are non-smokers. The probability of this is*

$$\begin{aligned}
\Pr(\textit{No smoker in the group}) &= \Pr(\overline{S} \cap \overline{S} \cap \overline{S} \cap \overline{S}) \\
&= \Pr(\overline{S}) \Pr(\overline{S}) \Pr(\overline{S}) \Pr(\overline{S}) \\
&= (2/3)(2/3)(2/3)(2/3) \\
&= \frac{16}{81}
\end{aligned}$$

Notice that the second line of the above formula utilized the result on independence, i.e., it assumed that the probability that one person in a group is a smoker is unaffected by whether or not the other individuals are smokers. Under this simplifying assumption, we find that

$$\begin{aligned}\Pr(\text{Smoking Table Needed}) &= 1 - \Pr(\text{Smoking Table Not Needed}) \\ &= 1 - 16/81 \\ &= \frac{65}{81} \\ &= 0.8025\end{aligned}$$

Under these assumptions, we see that the need for smoking tables far outstrips the proportion of smokers in the population! We may wish to investigate precisely how robust this conclusion is to variations on these assumptions.

Example 2.2 ***(The Probability of a Flush in Poker)*** *A flush in 5 card stud poker is obtained if all 5 cards are of the same suit, i.e., either 5 spades, 5 hearts, 5 diamonds, or 5 clubs. What is the probability of obtaining a flush?*

Solution 2.2 *The key to solving this problem is to ask, for each of the 5 cards, what is the probability of obtaining a card that will "keep alive" the possibility of obtaining a flush. When the first card is drawn,* any *card will leave alive the possibility of a flush. However, once the first card is drawn, the suit (spade, heart, diamond, club) of the flush (if one is to occur) has been determined. After the first card has been drawn, 51 cards remain in the deck, and 12 of them are in the suit of the first card. So the probability that the second card will leave alive the possibility of a flush is 12/51, or 4/17. If the second card leaves alive the possibility of a flush, only 11 cards of the correct suit remain in the deck out of the 50 cards that have not been drawn. So the probability that the third card will leave alive the possibility of a flush is 11/50. Continuing in this vein, we can see that the probability of a flush is*

$$\Pr(Flush) = (1)(12/51)(11/50)(10/49)(9/48) = \frac{33}{16\,660} = .001981$$

Example 2.3 ***(The Probability of Disease Transmission)*** *In epidemiology, we can consider two rather different but related probabilities, (a) the probability of disease transmission in a single isolated encounter, and (b) the probability of disease transmission occurring after several encounters. Suppose, for example, that a fatal disease is transmitted via handshake, but that the general probability of catching the disease if you shake hands with an infected person is only .10, and only 1 person in 10 in the population is infected. If we assume that propensity to shake hands is unrelated to presence or absence of the disease, this means that the probability of catching the disease after 1 handshake is only 1 in 100, i.e., .01. Suppose you are a politician who is planning to shake 10 hands randomly this afternoon. What is the probability that you will contract the disease?*

Solution 2.3 *The probability of catching the disease is much more difficult to compute directly than is the probability of not catching the disease. Not catching the disease involves not catching it on any of the handshakes. Let* T *represent the event that the disease is transmitted in a single handshake.* $\Pr(T) = .01$, *and* $\Pr(\overline{T}) = .99$. *In order not to catch the disease, you must have the event* $\overline{T}$ *occur on all 100 handshakes. We have*

$$\Pr(\text{No Transmission}) = \left(\frac{99}{100}\right)^{10} = \frac{90\,438\,207\,500\,880\,449\,001}{100\,000\,000\,000\,000\,000\,000}$$

and

$$\Pr(\text{Transmission}) = 1 - \left(\frac{99}{100}\right)^{10} = .09562$$

Example 2.4 ***(Rolling a 6-Straight in Dice)*** *Suppose you were to roll 6 fair dice (or one fair die six consecutive times). What is the probability that you would obtain a 6-straight, i.e., have the numbers 1,2,3,4,5,6 come up equally often?*

Solution 2.4 *The "keep it alive" sequential strategy works beautifully on problems like this. We simply ask what is the probability that the* 6*-straight will be "kept alive" on each die throw. What is the probability that the first throw will leave "alive" the possibility of a* 6*-straight? Of course it is 1, since whatever number is rolled will not rule out the possibility that subsequent throws of the die will produce other numbers. But once the first die is thrown, what is the probability that the second die will leave alive a* 6*-straight. Once the first die is thrown, there are* 5 *remaining numbers (out of* 6*). So the probability that the second throw will leave a* 6*-straight alive is* 5/6*. Once the second die is thrown, there are only* 4 *numbers left that will leave alive a* 6*-straight. So the probability that the third throw will leave alive a* 6*-straight, given that the first two throws did, is* 4/6*. Continuing in this manner, we see that the probability of a* 6*-straight is*

$$\begin{aligned} \Pr(6\text{–straight}) &= (1)(\frac{5}{6})(\frac{4}{6})(\frac{3}{6})(\frac{2}{6})(\frac{1}{6}) \\ &= \frac{5}{324} = .01543 \end{aligned}$$

Example 2.5 ***(The Phone Book Problem)*** *Suppose you open the phone book, and, without looking, select a page "at random." Then, without looking, you point at a line in the phone book, circle that phone number, and the next 12, for a total of 13 phone numbers. (Do you feel lucky?) What is the probability that there are, within that group of 13 numbers, at least two numbers with the same last two digits? (If there is, you win!!) The following* 4 *phone numbers are an example of a match.*

682-87**87**
547-9002
778-7891
666-54**87**

Solution 2.5 *We need to make a simplifying assumption, i.e., that all numbers from* 0 *to* 9 *are equally likely to occur in the last a phone number, and that they occur at random. We then make use of the "keep it alive" strategy and the Second Theorem of probability. There are 100 possible pairs of digits in the last* 2 *positions in a phone number. Of course the first number leaves alive the possibility of No Match. Once the first number has occurred, there are* 99 *possible digit pairs (out of* 100*) remaining that will not match the first. Once the first two numbers have occurred, there are* 98 *possible numbers remaining. We continue this way for 13 positions in* *the sequence. So the probability we seek is*

$$\begin{aligned}
\Pr(\text{Match}) &= 1 - \Pr(\text{No Match}) \\
&= 1 - \left(\frac{100}{100}\right)\left(\frac{99}{100}\right)\left(\frac{98}{100}\right)\left(\frac{97}{100}\right)\cdots\left(\frac{88}{100}\right) \\
&= 1 - \prod_{i=88}^{100}\left(\frac{i}{100}\right) \\
&= \frac{27208253884016789933}{48828125000000000000} = .557255
\end{aligned}$$

which is slightly more than 5/9. Your odds of winning this game are about 5 to 4, which, in Las Vegas, would be like a license to print money.

Example 2.6 ***(The Birthday Problem)*** *Suppose there are 23 people in a room, and they have assembled essentially at random with respect to birthdays. What is the probability that at least two people in the room have the same birthday?*

Solution 2.6 *C.P.*

3. Counting Rules and Combinatorics

The sequence ("keep it alive approach") works very well for a huge number of difficult problems. However, certain problems cannot be solved easily using this approach, but can be solved using the general rule for equally likely elementary events:

$$\Pr(A) = \frac{N_A}{N_\Omega}$$

where N_A is the number of elementary events in A, the event of interest, and N_Ω is the number of elementary events in the sample space Ω. The sample space is huge in most problems, and we cannot solve the problem by inspection. Rather, must construct, by *counting rules*, the values for the numerator and denominator of the above formula.

Combinatorics rules are difficult to master for many students, because they require a mode of thought that is foreign to us when we first encounter it. It may help to keep reminding yourself that counting rules are not (directly) about probability — they are about counting the number of ways you can produce arrangements that fit a particular description.

In the following sections, we examine 4 of the key rules of combinatorics. These rules can be used to solve (literally) thousands of complex and challenging problems.

3.1. The general rule for the number of sequences

The first rule of combinatorics is used to derive the other three rules. It is deceptively simple. Consider a sequence of events $E_i,\ i = 1, 2, \ldots k$. Let $N_i,\ i = 1, 2, \ldots k$ be the number of alternatives at the ith point in the sequence, conditional on everything that has gone before. Then the total number of possible sequences is

$$\prod_{i=1}^{k} N_i$$

Example 3.1 ***(Scheduling Courses)*** *You are trying to construct a schedule. You have 3 courses you could take at 8:30, and 2 courses you could take at 9:30. How many different possible schedules are there?*

Solution 3.1 *The answer is the product of 3 and 2, or 6. It is easy to see how this rule is derived, if you simply produce a diagram of the possibilities. Suppose that the available courses at 8:30 are labeled A, B, and C, and the available courses at 9:30 are D and E. You obtain 6 possible pathways, as shown on the following page..*

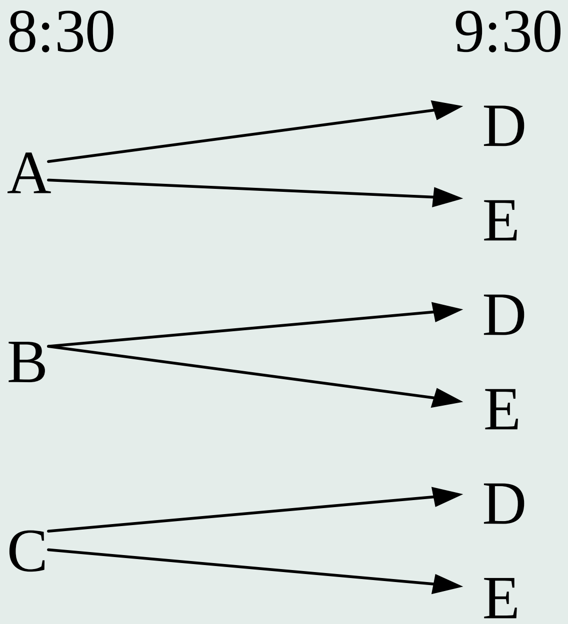

Figure 1: Diagramming the number of possible sequences.

3.2. Permutations

The number of *permutations*, or distinct orderings, of N objects is $N!$, the product of the integers counting down from N to 1, or

$$N! = N(N-1)(N-2)...(2)(1)$$

Example 3.2 *Compute* $3!$, $9!$, *and* $20!$

Solution 3.2

$$\begin{aligned} 3! &= (3)(2)(1) = 6 \\ 9! &= (9)(8)(7)(6)(5)(4)(3)(2)(1) = 362880 \\ 20! &= 2432\,902\,008\,176\,640\,000 \end{aligned}$$

Clearly, factorials get very large very quickly as a function of N.

Example 3.3 *There are* 7 *students in a seminar. Each student must give a presentation. How many distinctly different orderings of the* 7 *presentations are there?*

Solution 3.3 *The answer is* $7! = 5040$.

The derivation of the permutation rule is a straightforward consequence of the general rule for the number of sequences. Suppose you have just 3 items, A, B, C. How many distinctly different orderings can you construct? You have 3 possibilities at the first position. Once the first position is determined, you have only 2 possibilities for the second position. Once the first 2 positions are determined, there remains only one possibility for the third position. So there are $(3)(2)(1) = 6$ distinct orderings of 3 objects. The situation is diagrammed below

3.3. Permutations with selection

In some cases, we wish to select a group from a larger group, and then order them. How many ways can we select r objects from N objects and then order them? This is called "the number of permutations of N objects taken r at a time," and is denoted

$$
{}_N P_r = \underbrace{(N)(N-1)(N-2)\cdots(N-r+1)}_{r \ values}
$$

${}_N P_r$ is the product of the r integers counting down from N.

Lemma 3.1 ***Remark 3.1*** *Some books give the formula for ${}_N P_r$ as*

$$
{}_N P_r = \frac{N!}{(N-r)!}
$$

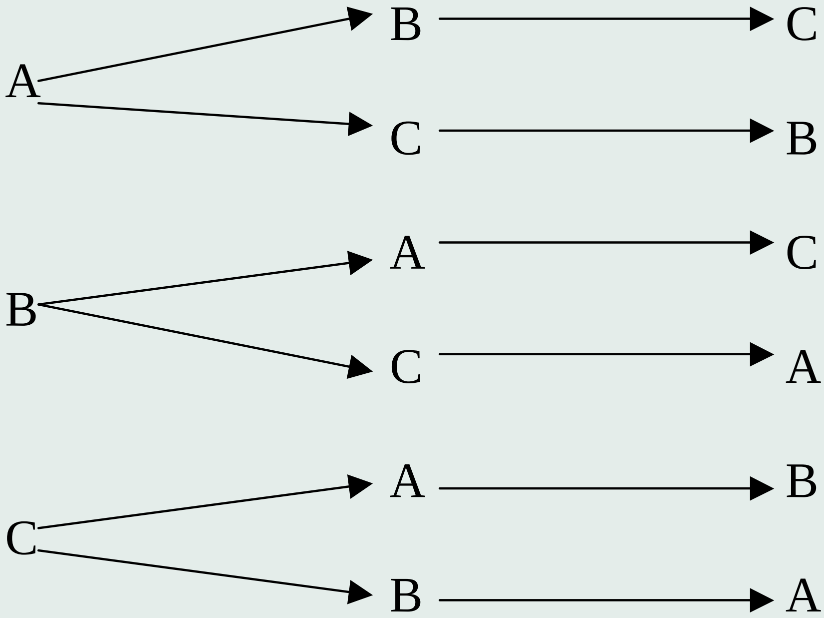

Figure 2: Diagramming the permutations of 3 objects.

This formula is not very efficient. To see why, consider the expression for ${}_6P_4$ *using the two alternate versions of the formula.*

$$\begin{aligned} {}_6P_4 &= (6)(5)(4)(3) \\ &= \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{(6)(5)(4)(3)(2)(1)}{(2)(1)} \end{aligned}$$

Example 3.4 *For example, suppose there are* 6 *people trying to get into my office, but I have only* 3 *chairs. How many ways can I select* 3 *people out of* 6*, and then order them into the* 3 *chairs?*

Solution 3.4 *The solution is*

$${}_6P_3 = (6)(5)(4) = 120$$

3.4. Combinations

In many situations, you are only interested in the number of ways you can select r objects from N objects, without respect to order. For example, suppose I had 3 tickets to a concert, and there were 6 students who wanted the tickets. How many different groups could I select to get the tickets? In this case, order doesn't matter. For example, if we label the 6 people A, B,C,D,E,F, the groups ABC and CBA are equivalent, since, for this analysis, the only thing that matters is who gets tickets! The number of combinations (different sets) of N objects taken r at a time is obtained by computing by the number of permutations, then dividing by the number of orders of the r objects, thereby "factoring out the order."

Several different notations are used for "N choose r," the number of combinations of N objects taken r at a time. Two common notations are

$$_N C_r$$

and

$$\binom{N}{r}$$

Remark 3.2 *The formula for the number of combinations is often given as*

$$\binom{N}{r} = \frac{N!}{(N-r)!r!} \tag{1}$$

This is often woefully inefficient. If we recall that this formula is equivalent to

$$\binom{N}{r} = \frac{_N P_r}{r!}$$

we realize that a more efficient way to compute combinations is the following: $\binom{N}{r}$ *is the ratio of the product of the* r *integers counting down from* N *divided by the product of the* r *integers counting up from* 1.

Remark 3.3 *The computation of combinations can be made even more efficient by remembering the following rule*

$$\binom{N}{r} = \binom{N}{N-r}$$

So when confronted with a combination problem, ask which number is smaller, r or $N-r$. Call this value k. Then compute the answer as $\binom{N}{k}$.

Example 3.5 *Compute $\binom{200}{198}$*

Solution 3.5 *Using the method described in the remark above, we realize that*

$$\binom{200}{198} = \binom{200}{2} = \frac{(200)(199)}{(2)(1)} = 19900$$

Remark 3.4 *If we had used Equation 1, and processed the formula literally, the computation would have been horribly inefficient, as we would have had to compute*

$$\frac{200!}{198!2!}$$

Example 3.6 ***(The Probability of a Flush, Revisited)*** *Compute the probability of a flush in 5 card stud poker, using the combinatorial approach.*

Solution 3.6 *The combinatorial approach to solving this problem is completely different from the "keep it alive" sequential approach.* N_Ω *is simply the number of distinctly different hands of* 5 *cards that can be drawn from a deck of* 52*. This is* ${}_{52}C_5$*.* N_A *is the number of different ways you can construct a flush. Consider only flushes in spades. There are 13 spades, so there are* ${}_{13}C_5$ *different spade flushes. There are equally many diamond, heart, and club flushes. So the probability of a flush must be*

$$\frac{4\binom{13}{5}}{\binom{52}{5}}$$

After some reduction, you can see that it is indeed equal to our previous answer.

$$\frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{\frac{4(13)(12)(11)(10)(9)}{(5)(4)(3)(2)(1)}}{\frac{(52)(51)(50)(49)(48)}{(5)(4)(3)(2)(1)}} = \frac{(52)(12)(11)(10)(9)}{(52)(51)(50)(49)(48)}$$

Example 3.7 *How many sets, including the null set, can be formed from N objects?*

Solution 3.7 *This is a classic example of a problem that can be solved with different approaches that yield formulas that are equivalent, but look completely different. Consider the simple case of $N = 3$. Call the 3 elements A, B, and C. We can ask "How many 0 element sets are there?" The answer, of course, is 1, the Null Set. Next, we might ask how many 1 element sets there are, and the answer is clearly 3. Each of these amounts cam be computed as a combination, i.e., ${}_3C_0 = 1$, ${}_3C_1 = 3$. So the total number of sets that can be formed from 3 elements is the sum of the number of 0 element,1 element, 2 element, and 3 element sets that can be selected from 3 objects. This is*

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

So, more generally, the number of sets that can be formed from an Ω with N elements is

$$\sum_{i=0}^{N} \binom{N}{i}$$

However, there is a much simpler answer, obtained in a completely different way. Simply realize that for each set formed from the N objects, there is a unique N digit binary code. Each element in Ω is coded either 0 or 1 *depending on whether it is in the set. For example, the following table lists all the sets that can be made from the objects A, B, C, and the associated binary code. Can you see from the table below why the answer must be* 2^N *?*

A	B	C	Set
0	0	0	∅
1	0	0	A
0	1	0	B
0	0	1	C
1	1	0	A,B
1	0	1	A,C
0	1	1	B,C
1	1	1	A,B,C