Notation Fundamentals

Definition. A *matrix* is defined as an ordered array of numbers, of dimensions p, q. Our standard notation for a matrix **A** of order p, q will be:

$$ {}_p\mathbf{A}_q \tag{1} $$

There are numerous other notations. For example, one might indicate a matrix of order p, q as **A** $(p \times q)$. Frequently, we shall refer to such a matrix as “a p by q matrix.”

On occasion, we shall refer explicitly to the *elements* of a matrix (i.e., the numbers or random variables in the array). In this case, we use the following notation to indicate that **A** is a matrix with elements a_{ij}.

$$\mathbf{A} = \left\{ a_{ij} \right\} \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1q} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2q} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pq} \end{bmatrix}$$

Definition *A column vector of numbers or random variables is a matrix of order* $p \times 1$. *We will, in general, indicate column vectors with the following notation:*

$$ {}_{p}\mathbf{a}_{1} \tag{3} $$

Definition A *row vector* of numbers or random variables is a matrix of order $1 \times q$. We will, in general, indicate row vectors with the following notation:

$$ {}_1\mathbf{a}'_q \tag{4} $$

An common alternate notation is

$$ {}_1\mathbf{a}^{\mathrm{T}}{}_q \tag{5} $$

A column vector with all elements equal to one will be symbolized as **1**.

Special Matrices

We will refer occasionally to special types of matrices by name. For any ${}_p\mathbf{A}_q$,

If $p \neq q$, **A** is a *rectangular* matrix.

If $p = q$, **A** is a *square* matrix.

In a *square* matrix, the elements a_{ii}, $i = 1, \; p$ define the *diagonal* of the matrix.

A square matrix is *lower triangular* if $a_{ij} = 0$ for $i < j$.

A square matrix is *upper triangular* if $a_{ij} = 0$ for $i > j$.

A square matrix is a *diagonal matrix* if $a_{ij} = 0$ for $i \neq j$

A square matrix is a *scalar matrix* if it is a diagonal matrix and all diagonal elements are equal.

An *identity matrix* is a scalar matrix with diagonal elements equal to one. We use the notation $\mathbf{I}_p$ to denote a $p \times p$ identity matrix.

0, a matrix composed entirely of zeros, is called a *null matrix*.

A square matrix is *symmetric* if $a_{ij} = a_{ji}$ for all $i,\ j$

A 1×1 matrix is a *scalar*.

Example *Some examples follow:*

A rectangular matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

A square matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

An upper triangular matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

A diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

A scalar matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

A symmetric matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

Some Matrix Operations

In this section, we review the fundamental operations on matrices.

Matrix (and Vector) Addition and Subtraction

For the addition and subtraction operations to be defined for two matrices **A**, **B**, they must be *conformable*.

Definition (Conformability for Addition and Subtraction). *Two matrices are conformable for addition and subtraction if and only if they are of the same order.*

Definition (Matrix Addition and Subtraction).

Let $\mathbf{A} = \left\{a_{ij}\right\}$ *and* $\mathbf{B} = \left\{b_{ij}\right\}$ *be two matrices that are conformable for addition. The sum* $\mathbf{C} = \mathbf{A} + \mathbf{B}$ *is defined as:*

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \left\{c_{ij}\right\} = \left\{a_{ij} + b_{ij}\right\} \quad (6)$$

The difference $\mathbf{D} = \mathbf{A} - \mathbf{B}$ *is defined as*

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \left\{ d_{ij} \right\} = \left\{ a_{ij} - b_{ij} \right\} \quad (7)$$

Comment. Matrix addition and subtraction are natural, intuitive extensions to scalar addition and subtraction. One simply adds elements in the same position.

Example Let $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 4 \\ 4 & 4 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

Find $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$.

Solution. $\mathbf{C} = \begin{bmatrix} 4 & 6 & 6 \\ 4 & 6 & 5 \\ 5 & 7 & 2 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$

Definition (Matrix Equality). Two matrices are equal if and only if they are of the same row and column order, and have all elements equal.

Matrix addition has some important mathematical properties, which, fortunately, mimic those of scalar addition and subtraction. Consequently, there is little “negative transfer” involved in generalizing from the scalar to the matrix operations.

For matrices **A**, **B**, and **C**, properties include:

Associativity

$$\mathbf{A}+\left(\mathbf{B}+\mathbf{C}\right)=\left(\mathbf{A}+\mathbf{B}\right)+\mathbf{C} \qquad (8)$$

Commutativity

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \qquad (9)$$

Scalar Multiples and Scalar Products

In the previous section, we examined some matrix operations, addition and subtraction, that operate very much like their scalar algebraic counterparts. In this section, we begin to see a divergence between matrix algebra and scalar algebra.

Definition (Scalar Multiple). *Given a matrix* $\mathbf{A} = \left\{ a_{ij} \right\}$ *, and a scalar c. Then* $\mathbf{B} = c\mathbf{A} = \left\{ ca_{ij} \right\}$ *is called a scalar multiple of* **A**.

Comment. Scalar *multiples* are not to be confused with *scalar products*, which will be defined subsequently. Scalar multiplication is a simple idea --- multiply a matrix by a scalar, and you simply multiply every element of the matrix by the scalar.

Example. $\textit{Let}\ \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$

$\textit{Then}\ 2\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 6 & 8 \end{bmatrix}.$

For matrices **A** and **B**, and scalars a and b, scalar multiplication has the following mathematical properties:

$$(a+b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$$

$$a\left(\mathbf{A}+\mathbf{B}\right) = a\mathbf{A} + a\mathbf{B}$$

$$a\left(b\mathbf{A}\right) = \left(ab\right)\mathbf{A}$$

$$a\mathbf{A} = \mathbf{A}a$$

Definition (Scalar Product). *Given row vector* ${}_1\mathbf{a}'_p$ *and* ${}_p\mathbf{b}_1$. *Let* $\mathbf{a}' = \{a_i\}$ *and* $\mathbf{b} = \{b_i\}$. *The scalar product* $\mathbf{a}'\mathbf{b}$ *is defined as*

$$\mathbf{a}'\mathbf{b} = \sum_{i=1}^{p} a_i b_i \qquad (10)$$

Note: This is simply the sum of cross products of the elements of the two vectors.

Example Let $\mathbf{a}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$. Then

$\mathbf{a}'\mathbf{b} = 11$.