Random Vectors and Random Matrices

In this section, we extend our results on linear combinations of variables to *random vector* notation. The generalization is straightforward.

Definition (Random Vector). A *random vector* ξ is a vector whose elements are random variables.

Definition (Random Matrix). A *random matrix* Ψ is a matrix whose elements are random variables.

Definition (Expected Value of a Random Vector or Matrix). The expected value of a random vector (or matrix) is a vector (or matrix) whose elements are the expected values of the individual random variables that are the elements of the random vector.

Example (Expected Value of a Random Vector). Suppose, for example, we have two random variables **x** and **y**, and their expected values are 0 and 2, respectively. If we put these variables into a vector $\boldsymbol{\xi}$, it follows that

$$E(\boldsymbol{\xi}) = E\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Random Matrices and Their Expected Values

A *random matrix* $\mathbf{\Psi}$ is a matrix of random variables. The *expected value* of a random matrix $\mathbf{\Psi}$ is the matrix of expected values of the random variables in $\mathbf{\Psi}$.

Result. (Variance-Covariance Matrix of a Random Vector). Given a random vector $\boldsymbol{\xi}$ with expected value $\boldsymbol{\mu}$, the variance-covariance matrix is defined as

$$\boldsymbol{\Sigma}_{\xi\xi} = E\left\{(\boldsymbol{\xi} - \boldsymbol{\mu})(\boldsymbol{\xi} - \boldsymbol{\mu})'\right\}$$

$$= E(\boldsymbol{\xi}\boldsymbol{\xi}') - \boldsymbol{\mu}\boldsymbol{\mu}'$$

If $\boldsymbol{\xi}$ is a deviation score random vector, then

$$\boldsymbol{\Sigma}_{\xi\xi} = E(\boldsymbol{\xi}\boldsymbol{\xi}')$$

(C.P.)

Matrix Expected Value Algebra

As a generalization of results we presented in scalar algebra, we find that, for matrices of constants $\mathbf{B}$ and $\mathbf{C}$, random vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, random matrix $\mathbf{X}$, and a vector of constants $\mathbf{c}$,

$$E(\mathbf{c}) = \mathbf{c} \tag{1}$$

$$E(\boldsymbol{\eta} + \boldsymbol{\xi}) = E(\boldsymbol{\eta}) + E(\boldsymbol{\xi}) \tag{2}$$

$$E(\mathbf{B}'\boldsymbol{\xi}) = \mathbf{B}'E(\boldsymbol{\xi}) \tag{3}$$

$$E(\mathbf{BXC}) = \mathbf{B}\,E(X)\,\mathbf{C} \tag{4}$$

Variance-Covariance Matrix of a Set of Linear Combinations

Theorem (Variance-Covariance Matrix of Linear Combinations). Given $\mathbf{x}$, a random vector with p variables, having variance-covariance matrix $\mathbf{\Sigma_{xx}}$. The variance-covariance matrix of any set of linear combinations $\mathbf{y} = \mathbf{B}'\mathbf{x}$ may be computed as

$$\mathbf{\Sigma_{yy}} = \mathbf{B}'\mathbf{\Sigma_{xx}}\mathbf{B} \tag{5}$$

Covariance Matrix for Two Sets of Linear Combinations

In a similar manner, we may prove the following:

Theorem (Covariance Matrix of Two Sets of Linear Combinations). Given $\mathbf{x}$ and $\mathbf{y}$, two random vectors with p and q variables having covariance matrix $\mathbf{\Sigma_{xy}}$. The covariance matrix of any two sets of linear combinations $\mathbf{w} = \mathbf{B}'\mathbf{x}$ and $\mathbf{m} = \mathbf{C}'\mathbf{y}$ may be computed as

$$\mathbf{\Sigma_{wm}} = \mathbf{B}'\mathbf{\Sigma_{xy}}\mathbf{C} \tag{6}$$