Model Equations and Model Constraints

When a model is falsifiable, it imposes *constraints* on data. That is, not all data can fit the model, so the data must satisfy some constraint conditions in order to fit the model.

Here is a simple example.

Suppose you have 3 data values, a, b, and c.
You have a model that says these data values can be explained in terms of 2 parameters, x and y, via the equations

$$x + y = a,$$
$$x - y = b,$$
$$2x = c$$

Elimination of Parameters

We eliminate the parameters from the system to see what it "tells us about the data."

We use the third equation to eliminate *x*. Since $x = c/2$, we can rewrite the first two equations as

$$c/2 + y = a$$
$$c/2 - y = b$$

Adding these two together, we get

$$c = a + b$$

This model has one degree of freedom, represented by this one constraint equation. **Only data sets satisfying this constraint equation can fit the model!**

Using Model Constraints to Solve for Model Parameters

Suppose a data set actually does fit the model. What is the solution for x and y?

Substituting $a+b$ for c in the original model equations, it is rather easy to see that

$$x = \frac{a+b}{2} = \frac{c}{2}, \quad y = \frac{a-b}{2}$$

Symbolic Algebra Software

Software like *Mathematica* (you can license it for about $50 a year) does such operations in an eyeblink. Example

```
Eliminate[
 {x + y == a, x - y == b, 2 x == c},
 {x, y}
]
```

```
c == a + b
```

You can do the same thing in factor analysis, retracing Spearman's steps.

$$\mathbf{f} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} \theta_5 & 0 & 0 & 0 \\ 0 & \theta_6 & 0 & 0 \\ 0 & 0 & \theta_7 & 0 \\ 0 & 0 & 0 & \theta_8 \end{pmatrix}$$

```
Eliminate[
  {θ1^2 + θ5 == σ1,1,
   θ1 θ2 == σ2,1,
   θ2^2 + θ6 == σ2,2,
   θ1 θ3 == σ3,1,
   θ2 θ3 == σ3,2,
   θ3^2 + θ7 == σ3,3,
   θ1 θ4 == σ4,1,
   θ2 θ4 == σ4,2,
   θ3 θ4 == σ4,3,
   θ4^2 + θ8 == σ4,4},
  {θ1, θ2, θ3, θ4, θ5, θ6, θ7, θ8}]
```

Mathematica returns

$$\sigma_{3,1}\,\sigma_{4,2} == \sigma_{3,2}\,\sigma_{4,1} \,\&\&\, \sigma_{2,1}\,\sigma_{4,3} == \sigma_{3,2}\,\sigma_{4,1}$$

The “&&” means “and” in *Mathematica* language.

These “tetrad difference equations” are the conditions that the data must satisfy to fit a 1-factor model. Notice that the variances of the variables ***never appear*** in these 4 equations!! What does that tell you? (C.P.)

Solving for Model Parameters

You can add these conditions to the original model equations and solve them.

```
Solve[
 {θ1^2 + θ5 == σ1,1,
  θ1 θ2 == σ2,1,
  θ2^2 + θ6 == σ2,2,
  θ1 θ3 == σ3,1,
  θ2 θ3 == σ3,2,
  θ3^2 + θ7 == σ3,3,
  θ1 θ4 == σ4,1,
  θ2 θ4 == σ4,2,
  θ3 θ4 == σ4,3,
  θ4^2 + θ8 == σ4,4,
  σ3,1 σ4,2 == σ3,2 σ4,1,
  σ2,1 σ4,3 == σ3,2 σ4,1
 },
 {θ1, θ2, θ3, θ4, θ5, θ6, θ7, θ8}]
```

$$\text{Solve}[\{\theta_1^2+\theta_5==\sigma_{1,1},\ \theta_1\theta_2==\sigma_{2,1},\ \theta_2^2+\theta_6==\sigma_{2,2},\ \theta_1\theta_3==\sigma_{3,1},\ \theta_2\theta_3==\sigma_{3,2},\ \theta_3^2+\theta_7==\sigma_{3,3},\ \theta_1\theta_4==\sigma_{4,1},\ \theta_2\theta_4==\sigma_{4,2},\ \theta_3\theta_4==\sigma_{4,3},\ \theta_4^2+\theta_8==\sigma_{4,4},\ \sigma_{3,1}\sigma_{4,2}==\sigma_{3,2}\sigma_{4,1},\ \sigma_{2,1}\sigma_{4,3}==\sigma_{3,2}\sigma_{4,1}\},\ \{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7,\theta_8\}]$$

$$\left\{\theta_5 \to \sigma_{1,1} - \frac{\sigma_{2,1}^2\ \sigma_{3,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}^3\ \sigma_{4,1}^2},\right.$$

$$\theta_6 \to \sigma_{2,2} - \frac{\sigma_{2,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}\ \sigma_{4,1}^2},$$

$$\theta_7 \to \sigma_{3,3} - \frac{\sigma_{3,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}\ \sigma_{4,1}^2},$$

$$\theta_8 \to -\frac{\sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}} + \sigma_{4,4},$$

$$\theta_1 \to \frac{\sigma_{2,1}\ \sigma_{3,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sigma_{3,2}^{3/2}\ \sigma_{4,1}},$$

$$\theta_2 \to \frac{\sigma_{2,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}\ \sigma_{4,1}},$$

$$\theta_3 \to \frac{\sigma_{3,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}\ \sigma_{4,1}},$$

$$\left.\theta_4 \to \frac{\sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}}\right\}\}$$

I printed above only one of the two solutions *Mathematica* returns. The other solution is the same, except all the factor loadings have the signs reversed.

$$\{\{\theta_5 \to \sigma_{1,1} - \frac{\sigma_{2,1}^2\ \sigma_{3,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}^3\ \sigma_{4,1}^2},$$

$$\theta_6 \to \sigma_{2,2} - \frac{\sigma_{2,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}\ \sigma_{4,1}^2},$$

$$\theta_7 \to \sigma_{3,3} - \frac{\sigma_{3,1}^2\ \sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}\ \sigma_{4,1}^2},$$

$$\theta_8 \to -\frac{\sigma_{4,2}\ \sigma_{4,3}}{\sigma_{3,2}} + \sigma_{4,4},$$

$$\theta_1 \to -\frac{\sigma_{2,1}\ \sigma_{3,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sigma_{3,2}^{3/2}\ \sigma_{4,1}},$$

$$\theta_2 \to -\frac{\sigma_{2,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}\ \sigma_{4,1}},$$

$$\theta_3 \to -\frac{\sigma_{3,1}\ \sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}\ \sigma_{4,1}},$$

$$\theta_4 \to -\frac{\sqrt{\sigma_{4,2}}\ \sqrt{\sigma_{4,3}}}{\sqrt{\sigma_{3,2}}}\},$$